PARTITIONS OF PRODUCTS

BY

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ABSTRACT. This paper extends some applications of a theorem of Halpern and Lauchli on partitions of products of finitary trees. The extensions are to weak infinite products of dense linear orderings, and ultrafilter preservation for finite product Sacks forcing.

Introduction. The Halpern-Lauchli (HL) theorem is proved by J. D. Halpern and H. Lauchli [4]. Another proof is given by Halpern [3]. The original purpose of the theorem was to give a model for set theory in which the Boolean prime ideal theorem is true while the axiom of choice is false. This was done by J. D. Halpern and A. Levy [5].

The Laver-Pincus (LP) theorem is a reformulation of the HL theorem due independently to R. Laver (1969, unpublished) and D. Pincus (1974). In this form the theorem has been applied to finite products of dense order types (Laver, 1969, unpublished), product Sacks forcing (J. Baumgartner, 1974, unpublished), and the analytic Ramsey theory of ω^ω (K. Milliken [7]). Milliken gives proofs of the Laver-Pincus theorem in [7] and [8]. He also gives a joint generalization of it and Ramsey's theorem.

This paper will strengthen the existing applications of the HL theorem as they pertain to models of set theory without the axiom of choice, products of dense linear orderings, and product Sacks forcing. In the course of the paper we will give two proofs of the LP theorem. The original proof is one which parallels that of the HL theorem. One of our proofs will proceed directly from the result of the HL theorem. The other proof will be along the lines of the original proof but will obtain a stronger result.

The paper is organized as follows. The LP theorem and other results on products of trees are stated in *tree form* in §1. They are restated in *matrix form* in §2 and the other tree product theorems are reduced to the matrix form of the LP theorem there. The first proof of the LP theorem, the one proceeding directly from the result of the HL theorem, is given in §3.

§4 contains a theorem which is a special case of a conjecture of Pincus [10] on a model of set theory without the axiom of choice. The results of Halpern [2], Halpern and Levy [5] and Pincus [9] suggest a way to proceed from a given model of set theory without the axiom of choice to a combinatorial theorem which implies that the Boolean prime ideal theorem holds in the model. In [10] a model is given

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which satisfies the principle of dependent choice but not the full axiom of choice. The combinatorial principle which implies the Boolean prime ideal theorem in this model involves the countable ordinal numbers, α. The case $\alpha = 0$ is an easy consequence of the HL theorem. The case $\alpha = 1$ is what we will prove here. It properly generalizes the HL theorem. The conjecture is open for higher α's. The problem of giving a model of set theory satisfying dependent choice and the Boolean prime ideal theorem but not full choice is solved in [11]. An affirmative answer to the conjecture of this section nevertheless would still have interesting consequences. For example one could then give a model of set theory with choice in which there is a nonprincipal ultrafilter μ on the power set of ω (ω is the set of natural numbers), a Vitali (or Banach Tarski) nonmeasurable set V, and a Lusin set L such that no well ordering of the real numbers can be defined using μ, V, L, or real numbers as parameters.

§5 presents a generalization to weak infinite products of one of Laver's original applications of the LP theorem. The generalization is made possible by an infinite version of the LP theorem given in §1. Otherwise the proof closely follows Laver's own (1969, unpublished). It appears with his permission.

§6 concerns the theory of product Sacks forcing. If L is the constructible universe and s is a Sacks real it was known in folklore that every subset of ω in $L(s)$ includes or is disjoint from an infinite constructible subset of ω. Baumgartner used the Laver-Pincus theorem to generalize this to $L(s_1, \ldots, s_d)$ where $s_1, \ldots, s_d$ are independent (product generic) Sacks reals. R. Solovay (unpublished) strengthened the folklore result by showing that any selective ultrafilter in L generates a selective ultrafilter in $L(s)$. In this section we prove a selective ultrafilter version of the LP theorem and use it to obtain the corresponding result for $L(s_1, \ldots, s_d)$.

1. Tree theorems in subtree form. The subtree form of the HL theorem and its variants seem to be most convenient for applications, but the somewhat stronger but less perspicacious matrix form has been the key to the proofs.

A finitary tree is a partially ordered set $(T, \leqslant)$ (more simply T) with least element in which every element has a finite nonempty set of immediate successors and a totally ordered set of predecessors. $T(n)$, the nth level of T is the set of elements of T which have exactly n predecessors. For us all trees are finitary and hence $T(n)$ is finite for all n.

$S \subset T$ is a *strong subtree* if S, under its inherited order, is a tree and there is a *level function* $f\colon \omega \to \omega$ satisfying $S(n) \subset T(f(n))$, and for each n,

$$(\forall x \in S \cap T(n))(\forall y \in T(n + 1))(\exists z \in S)[\, x < y \to y \leqslant z\,].$$

$S_i \subset T_i$ for $i \in I$ is a *level family* of strong subtrees if there is a single f which is a level function for each S_i, $i \in I$.

THEOREM 1 (LAVER-PINCUS). *Let T_i, $i < d \in \omega$, be trees. Let*

$$\bigcup_{n \in \omega} \prod_{i < d} T_i(n) = \bigcup_{j < p} P_j$$

for some $p \in \omega$. Then there is a level family of strong $S_i \subset T_i$ for $i < d$ and a $j < p$ such that $\bigcup_{n \in \omega} \prod_{i < d} S_i(n) \subset P_j$.

The other tree partition theorems in this section are generalizations of Theorem 1 to infinitely many trees. The most direct generalization would be to replace d with ω in Theorem 1. Unfortunately using the axiom of choice one can construct a counterexample as follows: For each $i \in \omega$, let T_i be the full binary tree. Topologize $\prod_{i \in \omega} T_i(n)$ by the Tychonoff product of discrete topologies. A standard construction, involving the axiom of choice, gives for each n a partition $\prod_{i \in \omega} T_i(n) = P_n \cup Q_n$ where neither P_n nor Q_n has a perfect subset. It follows that neither P_n nor Q_n includes a set of the form $\prod_{i \in \omega} A_i$ where $A_i \subset T_i(n)$ and $|A_i| \geq 2$. $P = \bigcup_{n \in \omega} P_n$ and $Q = \bigcup_{n \in \omega} Q_n$ form a counterexample to the $d = \omega$ generalization of Theorem 1.

A generalization of Theorem 1 to infinitely many trees must relax in some way the requirement that the S_i be strong subtrees or the requirement that $\bigcup_n \prod_{i \in \omega} S_i(n) \subset P_j$. Milliken [7] proposes a relaxation of the first requirement. He calls T *perfect* if

$$(\forall x \in T)[\, x \text{ has at least 2 incomparable successors}].$$

He conjectures that Theorem 1 holds with $d = \omega$ when the level family of strong subtrees is replaced by a level family of perfect subtrees. It is unknown whether Milliken's conjecture is true.[1] Nevertheless, Milliken and others have obtained many interesting consequences.

Our own results relax the requirement that $\bigcup_{n \in \omega} \prod_{i \in \omega} S_i(n) \subset P_j$. We only assert that some "tail" together with the given finite sequence is in P_j.

THEOREM 2. *Let T_i, $i \in \omega$, be trees. Let*

$$\bigcup_{n \in \omega} \prod_{i \in \omega} T_i(n) = \bigcup_{j < p} P_j$$

for some finite p. Then there is a level family (with level function f) of strong subtrees $S_i \subset T_i$, $i \in \omega$ and a $j < p$ such that

$$(\forall d \in \omega)(Ae\; n \in \omega)\Big(\forall \vec{x} \in \prod_{i < d} S_i(n)\Big)\Big(\exists \vec{y} \in \prod_{i > d} T_i(f(n))\Big)[\,\vec{x}\vec{y} \in P_j\,].$$

The expression "$(Ae\; x \in X)\ldots$" should be read "For all but finitely many $x \in X \ldots$" or "For almost every $x \in X \ldots$." The end to end join of the sequences $\vec{x}$ and $\vec{y}$ is written $\vec{x}\vec{y}$. In a slight abuse of notation this expression will be used both in the case of disjoint domains or in the case that Domain $\vec{x} = n$ and Domain $\vec{y} = m$ or ω.

Theorem 2 generalizes Theorem 1 in that for any d one could consider a partition which ignores X_i, $i \geq d$. If S_i, $i < d$ are produced as in Theorem 2 one could produce S_i, $i < d$ satisfying Theorem 1 by throwing away $S_i(n)$ for the finitely many levels n which do not satisfy

$$\Big(\forall \vec{x} \in \prod_{i < d} S_i(n)\Big)\Big(\exists \vec{y} \in \prod_{i > d} T_i(f(n))\Big)[\,\vec{x}\vec{y} \in P_j\,].$$

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[1]R. Laver has recently announced a proof of what we have called Milliken's conjecture. The original formulation of this conjecture may be due to J. Baumgartner.

Theorem 3 considers powers of a single tree T. If, in Theorem 2, all T_i are equal to T one still cannot expect all S_i to be equal to a fixed S. The problem is that equality types can separate elements. (Consider, for example, a partition of $\bigcup_{n \in \omega} T(n)^\omega$ given by $\vec{x} \in P_0 \leftrightarrow x_0 = x_1$ and $\vec{x} \in P_1 \leftrightarrow x_0 \neq x_1$.) What one says instead is that every equality type includes a more restrictive equality type in which the conclusion of Theorem 2 holds.

Let T be a fixed tree, $n, N \in \omega$, and $\vec{x} \in T(n)^N$. Type

$$\vec{x} = \left\{ \vec{y} \in \bigcup_{m \in \omega} T(m)^\omega \colon (\forall i, j < N) \left[x_i \leqslant y_i \wedge (x_i = x_j \leftrightarrow y_i = y_j) \right] \right\}.$$

If $\vec{x} \in T(n)^N$ and $\vec{y} \in T(m)^M$ say $\vec{x} \leqslant \vec{y} \leftrightarrow \text{Type } \vec{y} \subset \text{Type } \vec{x}$.

THEOREM 3. *Let T be a tree and $\vec{x} \in T(m)^M$ for some $m, M \in \omega$. Let $\text{Type } \vec{x} = \bigcup_{j < p} P_j$ for some finite p. Then there is a strong $S \subset T$ (with level function f), a $y \geqslant x$ with entries in S, and a $j < p$ such that*

$$(\forall d \in \omega)(Ae \, n \in \omega)(\forall \vec{w} \geqslant \vec{y} \colon \vec{w} \in S(n)^d)(\exists \vec{z} \in T(f(n))^\omega)\left[\vec{w}^\frown \vec{z} \in P_j \right].$$

To see that Theorem 3 generalizes Theorem 1 consider a single tree T formed from the disjoint union of the T_i, $i < d$ by adding a base point. $\vec{x}$ can be the sequence in $T(1)^d$ which picks the least element of each T_i. Notice that it is a rather strong assertion about S that it is both a strong subtree *and* has a $\vec{y} \geqslant \vec{x}$ with entries in S.

2. Tree theorems in matrix form.
The matrix form is the way the HL theorem was originally stated and it is the form in which it and most of its variants are proved.

$D \subset T$ is (h, k) *dense* if

$$(\exists x \in T(h))(\forall y \in T(h + k))\left[\exists z \in D \right]\left[x \leqslant y \rightarrow y \leqslant z \right].$$

x is called a *center of D*. Note that a minimal (h, k) dense set has a unique center. A $(0, k)$ dense set is called k *dense*. $M \subset \prod_{i < d} T_i$ is an (h, k) *matrix* if there are (h, k) dense $D_i \subset T_i$, $i < d$ such that $\prod_{i < d} D_i \subset M$. A vector of centers of the D_i is called a *center of M*. A $(0, k)$ matrix is called a k *matrix*.

THEOREM 2.1. *Let $\bigcup_n \prod_{i < d} T_i(n) = \bigcup_{j < p} P_j$ for some finite p. Either P_0 includes a k matrix for every k, or there is $j < p$ and $h \in \omega$ such that for every k, P_j includes an (h, k) matrix.*

Notice that Theorem 2.1 follows easily from the $p = 2$ case. In view of this subsequent theorems will be stated in terms of a 2 partition P, Q.

PROOF OF THEOREM 1 FROM THEOREM 2.1. This proof is quite straightforward. However, we will present it in some detail because some interesting points arise in the corresponding argument for Theorem 6.1 in §6.

According to Theorem 2.1 there is a single component of the partition, P, a single h, and for each k a level m_k such that $P \cap \prod_{i < d} T_i(m_k)$ includes an (h, k) matrix.

From this it follows that there is an infinite $X \subset \omega$ such that for each $m \in X$ if $s(X, m)$ is the successor in X of m, then $P \cap \prod_{i < d} T_i(s(X, m))$ includes an $(h, m + 1 - h)$ matrix. Such an X will be called a *set of tree levels* for the partition.

$\Pi_{i<d} \; T_i(h)$ is finite so for some $\vec{a} \in \Pi_{i<d} \; T_i(h)$, $Y = \{k \in X: (\exists m)[k = s(X, m) \wedge P \cap \Pi_{i<d} \; T_i(k)$ includes an $(h, m + 1 - h)$ matrix with center $\vec{a}]\}$ is infinite. Notice that for $m \in Y$, $s(X, m) \leqslant s(Y, m)$ so Y is also a set of tree levels for p. Furthermore the matrices from levels in Y have a common center. Such a Y will be called a *centered* set of tree levels for p. By changing h to the least element of Y, (stepping up one level and cutting down matrices) one gets that $Z = Y - \{$least element $Y\}$ is a centered set of tree levels for P, and furthermore $\vec{a} \in P$.

The centered set Z of tree levels gives rise to a level family $S_i \subset T_i$ of strong subtrees whose common level function is the increasing enumeration of Z and which satisfy $\bigcup_{k \in \omega} \Pi_{i<d} \; S_i(k) \subset P$. One takes a_i as the base of S_i. Having formed $S_i(k) \subset T_i(m)$ where m is the kth element of Z one lets A_i be the ith factor of the $(h, m + 1 - h)$ matrix centered at $\vec{a}$ included in $P \cap \Pi_{i<d} \; T_i(s(Z, m))$. A_i is $(h, m + 1 - h)$ dense so it contains a successor of every a_i successor at level $m + 1$ in T_i. In particular it contains a successor of every immediate T_i successor of an element of $S_i(k)$. Thus one takes

$$S_i(k + 1) = \left\{ b \in A_i : (\exists c \in S_i(k))[c < b] \right\}$$

and one will continue building full subtrees and have in addition

$$\prod_{i<d} S_i(k + 1) \subset P.$$

Theorem 2 and Theorem 3 have similar matrix forms. Let T_i, $i \in \omega$ be trees and $t: \omega \to \omega$ be a fixed nondecreasing function. An (h, k) *matrix* (with respect to t) is an $M \subset \Pi_{i<t(m)} \; T_i(m)$ for some m such that for some (h, k) dense $D_i \subset T_i(m)$, $i < t(h + k)$, every member of $\Pi_{i<t(h+k)} \; D_i$ extends to a member of M (so $m \geqslant h + k$). A *strong* (h, k) *matrix* (with respect to t) is an $M \subset \Pi_{i<t(m)} \; T_i(m)$ such that for some (h, k) dense D_i, $i < t(h + k)$,

$$\prod_{i<t(h+k)} D_i \times \prod_{t(h+k) \leqslant i < t(m)} T_i(m) \subset M.$$

A (strong) $(0, k)$ matrix is called a (strong) k *matrix*.

THEOREM 2.2. *Let T_i, $i \in \omega$ be trees and $t: \omega \to \omega$ be nondecreasing. If $\bigcup_n \Pi_{i<t(n)} \; T_i(n) = P \cup Q$ then either for some h and all k, Q includes a strong (h, k) matrix or for all k, P includes a k matrix.*

COROLLARY 2.2. *Under the assumptions of Theorem 2.2 if $\bigcup_n \Pi_{i<t(n)} \; T_i(n) = \bigcup_{j<p} P_j$ for some finite p, then for some $j < p$, $h \in \omega$ and all k, P_j includes an (h, k) matrix.*

Corollary 2.2 is an easy induction from the $p = 2$ case which, in turn, is weaker than Theorem 2.2. Theorem 2 follows from Corollary 2.2 as follows. For each n choose $\vec{y}_n \in \Pi_{i \geqslant n} \; T_i(n)$ and, given a partition of $\bigcup_n \Pi_{i \in \omega} \; T_i(n)$ define a partition of $\bigcup_n \Pi_{i<n} \; T_i(n)$ which takes $\vec{x} \in \Pi_{i<n} \; T_i(n)$ to the cell of $\vec{x}^{\frown}\vec{y}_n$. Apply Corollary 2.2 with $t(n) = n$ and form the S_i as in the proof of Theorem 1 letting $S_i(k + 1)$ come from an $(h, f(k) + 1)$ matrix in $\Pi_{i<m} \; T_i(m)$ for some appropriate m. Notice that, in the statement of Theorem 2, the $\vec{y} \in \Pi_{i \geqslant d} \; T_i(f(n))$ such that $\vec{x}^{\frown}\vec{y} \in P_j$ can actually be made to come from $\Pi_{i \geqslant d} \; S_i(n)$.

PROOF OF THEOREM 2.2 FROM THEOREM 2.1. Let $\bigcup_n \coprod_{i<t(n)} T_i(n) = P \cup Q$ as in Theorem 2.2. Assume for some k_0 that P includes no k_0 matrix. Partition $\bigcup_n \coprod_{i<t(k_0)} T_i(n)$ as follows.

$$\vec{x} \in Q^* \cap \prod_{i<t(k_0)} T_i(n) \leftrightarrow n < t(k_0) \vee \left(\forall \vec{y} \in \prod_{t(k_0)<i<t(n)} T_i(n)\right)[\vec{x}\vec{y} \in Q],$$

$$\vec{x} \in P^* \cap \prod_{i<t(k_0)} T_i(n) \leftrightarrow n \geqslant t(k_0) \wedge \left(\exists \vec{y} \in \prod_{t(k_0)<i<t(n)} T_i(n)\right)[\vec{x}\vec{y} \in P].$$

Theorem 1 applies to P^* and Q^*. If P^* includes a k_0 matrix it follows from the definition of P^* that P includes a k_0 matrix. Since we assumed otherwise P cannot include k matrices for all k. Therefore Q^* includes (h, k) matrices for some h and all k. If k is sufficiently large these matrices are at levels higher than $t(k_0)$. In this case the definition of Q^* shows that each such matrix gives rise to a strong (h, k) matrix in Q.

To obtain the matrix form of Theorem 3 one must reformulate the notion of matrix so as to take account of equality types.

Let T be a tree and $t: \omega \to \omega$ be nondecreasing. If $\vec{x} \in T(h)^{t(h)}$ let Type $\vec{x} = \{\vec{y} \geqslant \vec{x}: \vec{y} \in \bigcup_{m,n} T(n)^m\}$. This represents a slight change from the meaning of Type $\vec{x}$ in §1. Let $\vec{x}_0 \in T(h_0)^{t(h_0)}$ be fixed. $M \subset T(m)^{t(m)}$ is an (h, k) *matrix*, in Type x_0, $h \geqslant h_0$ if there is an $h + k$ dense set $D \subset T(m)$ and an $\vec{x}_1 \geqslant \vec{x}_0$ in $T(h)^{t(h)}$ such that every element of Type $\vec{x}_1 \cap D^{t(h+k)}$ extends to an element of M. M is a *strong* (h, k) matrix in Type $\vec{x}_0$ if it contains every extension of an element of Type $\vec{x}_1 \cap D^{t(h+k)}$. A (strong) k matrix is a (strong) (h_0, k) matrix.

THEOREM 2.3. *Let T be a tree, $t: \omega \to \omega$ be nondecreasing, and $\vec{x}_0$ be a fixed element of $T(h_0)^{t(h_0)}$. If Type $\vec{x}_0 = P \cup Q$ then either for some h and all k, Q includes a strong (h, k) matrix or for all k, P includes a k matrix.*

COROLLARY 2.3. *Under the assumptions of Theorem 2.3 if Type $x_0 = \bigcup_{j<p} P_j$ for some finite p, then for some $j < p$, $h \in \omega$, and all k, P_j includes an (h, k) matrix.*

It is left to the reader to derive Theorem 3 from Corollary 2.3.

PROOF OF THEOREM 2.3 FROM THEOREM 2.1. We may assume without loss of generality that $\vec{x}_0 \in T(1)^{t(1)}$. Furthermore, since a matrix is a subset of some $T(m)^{t(m)}$, we may restrict attention to Type $\vec{x}_0 \cap \bigcup_{n \in \omega} T(n)^{t(n)}$. For proof by contradiction, assume that P does not include a k_0 matrix in Type $\vec{x}_0$, some $k_0 \in \omega$. Let $N = |T(k_0^+)|$ and let

$$g: N \overset{\text{bijection}}{\to} T(k_0^+).$$

We define trees T_i, $i \leqslant N$, and apply Theorem 2.1 to $\bigcup_n \coprod_{i<N} T_i(n)$. For $i < N$, T_i is the subtree of T consisting of g_i, the successors of g_i, and O_T, the root of T. T_N is the tree such that

$$T_N(0) = O_T, \quad T_N(1) = \{f : f: t(k_0^+) \to N \text{ and } g \circ f \in \text{Type}(\vec{x}_0)\},$$

and in a harmless stretching of the notion of tree we let

$$T_N(k) = T_N(1) \quad \text{for } k > 1.$$

$$f_1 \leqslant_{T_N} f_2 \quad \text{iff } f_1 = O_T \text{ or } f_1 = f_2.$$

(Note: $\prod_{i<N} T_i(1) = \{(\vec{g}, f): f: t(k_0^+) \to N \text{ and } \vec{g} \circ f \in \text{Type } \vec{x}_0\}$.)

Partition $\bigcup_n \prod_{i<N} T_i(n) = P^* \cup Q^*$ by

$$(\vec{b}, f) \in Q^* \text{ iff every appropriate extension of } \vec{b} \circ f \text{ is in } Q.$$

$$(\vec{b}, f) \in P^* \text{ iff some appropriate extension of } \vec{b} \circ f \text{ is in } P.$$

($\vec{x} \in \bigcup_{m,n} T(n)^m$ is an appropriate extension of $\vec{b} \circ f$ if $\vec{x}|t(k_0^+) = \vec{b} \circ f$.)

We show

(1) A includes a 1 matrix in $\prod_{i<N} T_i(m) \leftrightarrow \Psi(A)$ is a k_0 matrix of Type $\vec{x}_0$ in $T(k_0 + m)^{t(k_0^+)}$ where $\Psi(A) = \{\vec{b} \circ f: (\vec{b}, f) \in A\}_0$.

PROOF. First show $(\vec{b}, f) \in \prod_{i<N} T_i(m)$ implies $\vec{b} \circ f \in \text{Type } \vec{x}_0$. $\vec{g} \circ f$ is of type $\vec{x}_0$ by the definition of T_N. Since $(\vec{b}, f) \in \prod_{i<N} T_i(m)$, $\vec{b}_i \geqslant g_i$ for all i. So for all i

$$(\vec{b} \circ f)_i \geqslant (\vec{g} \circ f)_i \geqslant \vec{x}_{0i}.$$

Furthermore if $\vec{x}_{0i} = \vec{x}_{0j}$ then $(\vec{g} \circ f)_i = (\vec{g} \circ f)_j$. Since $\vec{g}$ is a bijection $f_i = f_j$ and conversely so

$$(\vec{b} \circ f)_i = (\vec{b} \circ f)_j \quad \text{iff} \quad \vec{x}_{0i} = \vec{x}_{0j}.$$

Suppose A is a 1 matrix in $\prod_{i<N} T_i(1)$. (We take $m = 1$ for ease of exposition and with only slight loss of generality.) Then $A = \prod_{i<N} T_i(1)$. Let $D = T(k_0^+)$. We show $\{\vec{g} \circ f: f \in T_N(1)\} = D^{t(k_0^+)} \cap \text{Type } \vec{x}_0$.

The previous proof gives us

$$\Psi(A) = \{\vec{g} \circ f: f \in T_N(1)\} \subset D^{t(k_0^+)} \cap \text{Type } \vec{x}_0.$$

Let $\vec{x} \in D^{t(k_0^+)} \cap \text{Type } \vec{x}_0$. Let $f = \vec{g}^{-1} \circ \vec{x}$. Then $\vec{g} \circ f = \vec{x} \in \text{Type } \vec{x}_0$. So $f \in T_N(1)$, so $\vec{x} = \vec{g} \circ f \in \Psi(A)$. So

$$D^{t(k_0^+)} \cap \text{Type } \vec{x}_0 \subset \Psi(A).$$

Now suppose $\Psi(A)$ is a k_0-matrix of Type $\vec{x}_0$ in $T(k_0 + m)^{t(k_0^+)}$. Let $(\vec{b}, f) \in A$. Then $\{\vec{b}\}$ is 1 dense in $\prod_{i<N} T_i(m)$. So $\{(\vec{b}, f): f \in T_N(1)\}$ is a 1 dense matrix included in A.

Show

(2) A is an (h, k) matrix in $\prod_{i<N} T_i(m)$ implies that $\Psi^*(A)$, the set of all appropriate extensions of elements of $\Psi(A)$, includes a strong $(k_0 + h, k)$ matrix of Type $\vec{x}_0$ in $T(m + k_0)^{t(m+k_0)}$.

PROOF. Suppose A is as above and assume $h > 0$. Then there exists $\vec{a} \in \prod_{i<N} T_i(h)$ and (h, k) dense sets $D_i \subset T_i$ with a_i as the center such that for some $f \in T_N(1)$

$$\prod_{i<N} D_i \times \{f\} \subset A.$$

D_i is (h, k) dense in T_i means that D_i is $(h + k_0, k)$ dense in T. Let $D = \bigcup_{i<N} D_i$. Extend D in a conservative way to an $(k_0 + h + k)$ dense subset of $T(m + k_0)$. Let

$D_1 = T(m + k_0) - \{x: \exists i, x \geqslant a_i\}$. Then $D \cup D_1$ is an $(k_0 + h + k)$ dense subset of $T(m + k_0)$ and

$$(D \cup D_1)^{t(k_0^+)} \cap \mathrm{Type}(\vec{a} \circ f) \subseteq D^{t(k_0^+)} \cap \mathrm{Type}(\vec{a} \circ f).$$

Show $D^{t(k_0^+)} \cap \mathrm{Type}(\vec{a} \circ f) \subseteq \Psi(A)$:

For $\vec{x} \in D^{t(k_0^+)} \cap \mathrm{Type}(\vec{a} \circ f)$ define $g': N \to D$ such that $g_i' \geqslant a_i$ and $g_{f(j)}' = x_j$, all $j \in t(k_0^+)$. These conditions are consistent since $x_j \geqslant a_{f(j)}$. Then $\vec{g}' \in \prod_{i<N} D_i$ and $\vec{g}' \circ f \in \mathrm{Type}(\vec{a} \circ f)$ (since $\vec{a}$ and $\vec{g}'$ are both 1-1) and $\vec{g}' \circ f = \vec{x}$. So we have $(D \cup D_1)^{t(k_0^+)} \cap \mathrm{Type}(\vec{a} \circ f) \subseteq \Psi(A)$. From this it follows that every extension in $T(m + k_0)^{t(m + k_0)}$ of elements of $(D \cup D_1)^{t(k_0+h+k)} \cap \mathrm{Type}(\vec{a} \circ f)$ is in $\Psi^*(A)$. Furthermore $\vec{a} \circ f \geqslant \vec{x}_0$, so $\Psi^*(A)$ includes a strong $(k_0 + h, k)$ matrix in $T(m + k_0)^{t(m + k_0)}$.

PROOF OF THEOREM. Since P does not contain a k_0 matrix of Type $\vec{x}_0$ it follows that P^* does not include a 1 matrix. Therefore by Theorem 1 there is an h such that for all k, Q^* includes an (h, k) matrix. Therefore by (2) and the definition of Q^*, there is an h such that for all k, Q includes a strong (h, k) matrix of Type $\vec{x}_0$.

This concludes the proof of Theorem 2.3 from Theorem 2.1. Notice that Theorems 1, 2, 3, 2.2, and 2.3 have all been proven modulo a proof of Theorem 2.1.

3. Proof of Theorem 2.1 from the HL theorem. As previously mentioned, Milliken gives a proof of Theorem 2.1 in [8]. Thus we will content ourselves with presenting the basic ideas of the derivation of Theorem 2.1 from the HL theorem.

THEOREM (HALPERN-LAUCHLI). *Let T_i, $i < d < \omega$, be trees, and let $\prod_{i<d} T_i = \bigcup_{j<p} P_j$ for some $p \in \omega$. Then either P_0 includes a k matrix for every k, or there is $j < p$ and $h \in \omega$ such that for every k, P_j includes an (h, k) matrix.*

Notice that the Laver-Pincus theorem differs from the Halpern-Lauchli theorem in that the latter is concerned with partitions of the full products of the trees whereas the former is concerned with partitions of the union of products of the levels.

In stating our principles we will limit ourselves to length 2 sequences of trees T_1, T_2 in order to ease their digestion by the reader. The principles themselves are not dependent on the length of the sequence and in fact hold even if the sequences are infinite. On the other hand the presentation is not eased by limiting the partitions to 2-partitions. Consequently we will allow arbitrary finite partitions $\vec{Q}$ (a partition, for us, will be a sequence of sets rather than a family of sets).

DOMINATION PRINCIPLE (SIMPLE VERSION). *If B_1, $C_1 \subset T_1$ and B_2, $C_2 \subset T_2$ and every element of B_i has a successor in C_i, for $i = 1, 2$ (write $B_i \leqslant C_i$) and $\vec{Q}$ partitions $C_1 \times C_2$, then there is a partition $\vec{Q}'$ of $B_1 \times B_2$ with the same length as $\vec{Q}$ such that for all h, k, α, if $\vec{Q}_\alpha'$ includes an (h, k) matrix then so does $\vec{Q}_\alpha$ (i.e., $\vec{Q}$ is as good as $\vec{Q}'$).*

PROOF. Let $f_1: B_1 \to C_1$ and $f_2: B_2 \to C_2$ such that $x \leqslant f_j(x)$ (all $x \in B_j, j = 1, 2$). Define $\vec{Q}'$ by $\vec{Q}_\alpha' = \{(x_1, x_2): (f_1(x_1), f_2(x_2)) \in \vec{Q}_\alpha\}$ for all α in the domain of $\vec{Q}$. It is an easy matter to verify that $\vec{Q}$ is as good as $\vec{Q}'$.

DOMINATION PRINCIPLE (EXTENDED VERSION). *Let $\{B_i\}_{i \in \omega}$ be a sequence of subsets of T_1, $\{C_i\}_{i \in \omega}$ a sequence of subsets of T_2 such that for all $i < j$, $B_i \leqslant B_j$ and $C_i \leqslant C_j$. Let $B \subset T_1$, $C \subset T_2$ such that $B \leqslant \bigcup_i B_i$, $C \leqslant \bigcup_i C_i$.*

If $\vec{Q}$ is a partition of $\bigcup_i (B_i \times C_i)$ then there is a partition $\vec{Q}'$ of $B \times C$ such that $\vec{Q}$ is as good as $\vec{Q}'$.

PROOF. For each i, let $f_1^{(i)}$, $f_2^{(i)}$ be partial maps from B, C respectively to B_i, C_i respectively such that the domain of $f_1^{(i)}$, $(f_2^{(i)})$ is the largest subset of B, (C) which is dominated by B_i, (C_i) and such that $x \leqslant f_1^{(i)}(x)$ all $x \in \mathrm{Dom}\, f_1^{(i)}$ $(y \leqslant f_2^{(i)}(y)$ all $y \in \mathrm{Dom}\, f_2^{(i)})$. Let $f^{(i)}$ be the map on $B \times C$ induced by $\langle f_1^{(i)}, f_2^{(i)} \rangle$ and let $\{\langle x_n, y_n \rangle\}_{n \in \omega}$ be an enumeration of $B \times C$ (x_n denotes the 1st cordinate of the nth point, y_n denotes the second). We define a sequence of subsets of ω by recursion. Let $S_n = \{i \in S_{n-1} : f^{(i)}(x_n, y_n) \in \vec{Q}_{L_n}\}$ where $S_{-1} = \omega$ and L_n is chosen (take L_n to be the least such) so that S_n is infinite. In this way we get a sequence $S_0 \supseteq S_1 \supseteq \dots$ of infinite sets such that if $i, j \in S_n$ then $f^{(i)}(\langle x_n, y_n \rangle)$ and $f^{(j)}(\langle x_n, y_n \rangle)$ are in the same part of $\vec{Q}$. Define $\vec{Q}'$ by $\langle x_n, y_n \rangle \in \vec{Q}'_L$ if $f^{(i)}(\langle x_n, y_n \rangle) \in \vec{Q}_L$ for some (equiv. all) $i \in S_n$. We show that $\vec{Q}$ is as good as $\vec{Q}'$. Suppose $A_1 \times A_2 \subset Q'_L$ where A_j is (h, k) dense in T_j, $j = 1, 2$. Let n be such that each point of $A_1 \times A_2$ occurs among the first n points in the enumeration of $B \times C$. Then $f^{(i)}(\langle x, y \rangle) \in Q_L$ all $i \in S_n$, all $x \in A_1$, all $y \in A_2$ i.e., $\langle f_1^{(i)}(x), f_2^{(i)}(y) \rangle \in Q_L$ all $i \in S_n$, all $x \in A_1$, all $y \in A_2$. Since $x \leqslant f_1^{(i)}(x)$ and $y \leqslant f_2^{(i)}(y)$ we have $f_1^{(i)}[A_1]$ is (h, k) dense in T_1 and $f_2^{(i)}[A_2]$ is (h, k) dense in T_2. Hence $f_1^{(i)}[A_1] \times f_2^{(i)}[A_2]$ is an (h, k) matrix included in Q_L. Thus $\vec{Q}$ is as good as $\vec{Q}'$.

For application of the domination principle, extended version, take $T_1(i)$ for B_i, T_1 for B, $T_2(i)$ for C_i and T_2 for C. We obtain that any partition of $\bigcup_n (T_1(n) \times T_2(n))$ is as good as some partition of $T_1 \times T_2$. Now the Halpern-Lauchli theorem gives the conclusion of Theorem 2.1.

4. Finite matrix versions and a model of set theory. The HL theorem was originally applied in exhibiting a model for set theory in which the Boolean prime ideal theorem is true while the axiom of choice is false. This application does not use the version of the theorem stated in §3. Instead it uses a finite version, analogous to the finite version of Ramsey's theorem. We state and prove a finite version of Theorem 2.3.

THEOREM 4. *Let T, t, and $\vec{x}_0$ be as in* Theorem 3. *Fix p, $k \in \omega$. There is an $m \in \omega$ such that whenever* Type $\vec{x}_0 \cap T(m)^{t(m)} = \bigcup_{j < p} P_j$ *there are $j < p$ and $h < m$ such that P_j includes an (h, k) matrix.*

PROOF (ASSUMING COROLLARY 2.3). If Theorem 4 is false then for some p, k every Type $\vec{x}_0 \cap T(m)^{t(m)}$ has a partition into p pieces which fails to include an (h, k) matrix. Putting these partitions together to a single partition of Type $\vec{x}_0 \cap \bigcup_m T(m)^{t(m)}$ gives a contradiction to Corollary 2.3.

In [10] Pincus gives a model of set theory which satisfies the principle of dependent choice and the linear ordering theorem (every set can be linearly ordered) but does not satisfy full choice. He conjectures that this model satisfies the

Boolean prime ideal theorem. If this is true, a related model with choice would have the definability properties given in the introduction to this paper.

We state below a combinatorial conjecture involving the countable ordinals α which would imply the Boolean prime ideal theorem in the model. We then prove the $\alpha = 1$ case from Theorem 4.

Functions of order α are defined by induction on the countable ordinals, α. A function of order 0 is a map from ω to 2. A function of order $\alpha + 1$ is a map from ω into the set of functions of order α. A function of limit order β is a function on β whose value at $\alpha < \beta$ is a function of order α. The Cohen conditions of [10] are finite systems of equations (henceforth called *systems*) among such functions. A precise formulation of this notion is set forth below.

Let 0 and 1 be terms of order -1. For each countable ordinal α single out an infinite set of *function symbols* of order α. A *term* is 0, 1, or a finite sequence beginning with a function symbol. If $S \neq 0$, 1 is a term of order 0 or $\alpha + 1$ then Sn, $n \in \omega$ is a term of order $\alpha - 1$. If S is a term of limit order β then $S\alpha$, $\alpha < \beta$ is a term of order α. All terms are generated from function symbols by the above rules. A system $(U, \equiv)$ (more simply U) is a finite set of terms together with an equivalence relation between them. The following rules express the idea that equivalent terms are those connected by an equality and inequivalent terms are those connected by an inequality.

(a) If $S \in U$, so is every nonzero initial segment of S.

(b) If $S \equiv T$ then S and T have the same order.

(c) If f and g are distinct function symbols then $f \not\equiv g$.

(d) $0 \not\equiv 1$, and if $S \in U$ has order -1 then $S \equiv 0$ or $S \equiv 1$.

(e) If $S \equiv T$ and for some sequence W, $S^\frown W \in U$ then $T^\frown W \in U$ and $S^\frown W \equiv T^\frown W$.

U strongly extends V if $U \subset V$ and $\equiv_v$ restricts to $\equiv_u$ on U. If F is a set of function symbols $U|F$ is the system formed by restricting $\equiv_u$ to the set of terms beginning with function symbols from F (together with 0 and 1). $V \leqslant U$ if U strongly extends some system V^* which is obtained from V by a 1-1 order preserving substitution of function symbols in its terms. An α *system* is a system all of whose function symbols have order α. An α *type* is an α system with exactly one function symbol. If τ is an α type and f is a function symbol of U f satisfies τ when $\tau \leqslant U|\{f\}$.

Let U_n, $n \in \omega$, be an increasing sequence of α systems. If f is a function symbol of U_n, f^+ is the set of all function symbols g of U_{n+1} such that g satisfies the type $U_n|\{f\}$. A set of function symbols M of U_m is an n *matrix* when $U_{n+1}|f^+ = U_m|M$. f is the base of M.

Types and systems, modulo equivalence under 1-1 order preserving substitutions, are partially ordered by $\leqslant$. Recall that a subset D of a partially ordered set is dense when every element of the set bears $\leqslant$ to an element of D.

CONJECTURE. For each countable ordinal α there is a dense set D of systems. D together with the dense set D_0 of types occurring in D satisfy the following.

Let $\tau \in D_0$ and let U_n, $n \in \omega$, be an increasing sequence from D. There is an

$m \in \omega$ such that whenever $\{f \in U_m: f \text{ satisfies } \tau\} = P \cup Q$ either P or Q includes an n matrix, $n < m$, whose base satisfies τ.

PROOF OF THE CONJECTURE FOR $\alpha = 1$. Let D be the set of 1 systems U satisfying the following.

(1) There are N_0, $N_1 \in \omega$ such that $fpq \in U$ exactly when $f \in U$, $p < N_1$ and $q < N_0$.

(2) If $S \not\equiv T$ are terms of order $i \in 2$ then $Sp \not\equiv Tp$ for some $p \in N_i$. This says that the equivalence class of S is completely determined by the function $p \mapsto [Sp]$ from N_i into the order $i - 1$ equivalence classes.

(3) Every function from N_i into the order $i - 1$ equivalence classes is realized as one of the functions $p \rightarrow [Up]$. (See (2) above.)

Let U_n, $1 \leqslant n \in \omega$, be a sequence from D. Without loss of generality it can be assumed that the given type τ has the form $U_i|\{f\}$. Let T be the tree whose nth level is the set of functions from $N_0(U_n)$ to 2. Let $t: \omega \rightarrow \omega$ be defined by $t(n) \leqslant N_1(U_n)$. $x \in T(n)$ corresponds to $[S(x)] \in U_n$ where $S(x)q \equiv x(q)$ for all $q < N_0$. $\vec{x} \in T(n)^{t(n)}$ corresponds to the unique function symbol f of U_n satisfying $fp \equiv S(x)p$ for all $p < N_1$. This correspondence carries the type τ to a Type $\vec{x}_0$ for some $\vec{x}_0 \in T(1)^{t(1)}$. Similarly it carries n matrices of U_n to $n, 1$ matrices of Type $\vec{x}_0 \cap T(n)$. Thus this case of the conjecture is reduced to Theorem 4.

5. Products of the rational order type. Laver's original application of the LP theorem is the following where η is the order type of the set $\mathbf{Q}$ of rational numbers.

THEOREM (LAVER, 1969, UNPUBLISHED). *Let* $\mathbf{Q}^d = \bigcup_{j<p} P_j$. *There are* $X_i \subset \mathbf{Q}$, $i < d$, *of type* η *and a* $j < p$ *such that*

$$(Ae \; x_0 \in X_0)(Ae \; x_1 \in X_1) \cdots (Ae \; x_{d-1} \in X_{d-1})[\vec{x} \in P_j].$$

This section is devoted to obtaining a generalization of Laver's theorem to weak infinite powers of $\mathbf{Q}$. Theorem 2 plays approximately the same role in the generalization as Theorem 1 plays in obtaining Laver's theorem.

$\mathbf{Q}^{\underline{\omega}}$, the weak infinite power of $\mathbf{Q}$, is the set of all $\vec{x} \in \mathbf{Q}^\omega$ which are eventually 0. It follows immediately from Laver's theorem that if $\mathbf{Q}^{\underline{\omega}} = \bigcup_{j<p} P_j$, $d \in \omega$, and $\vec{u} \in \mathbf{Q}^{\underline{\omega}}$ then there are $j < p$ and $X_i \subset \mathbf{Q}$, $i < d$, of type η such that

$$(Ae \; x_{d-1} \in X_{d-1})(Ae \; x_{d-2} \in X_{d-2}) \cdots (Ae \; x_0 \in X_0)[\vec{x} \hat{\;} \vec{u} \in P_j].$$

Theorem 5 is in the direction of picking X_i, $i \in \omega$ and $j < p$ which works for each d and some U_d. What actually occurs is that there is a finite set U_d of possible vectors $\vec{u}$.

THEOREM 5. *Let* $\mathbf{Q}^{\underline{\omega}} = \bigcup_{j<p} P_j$. *There are* X_i, $i \in \omega$, *of type* η, *finite* $U_d \subset \mathbf{Q}^{\underline{\omega}}$, $d \in \omega$, *and* $j < p$ *such that*

$$(\forall d \in \omega)(Ae \; x_{d-1} \in X_{d-1}) \cdots (Ae \; x_0 \in X_0)(\exists \vec{u} \in U_d)[\vec{x} \hat{\;} \vec{u} \in P_j].$$

It would be interesting to know if U_d can be assumed to be a singleton. It would also be interesting to know if the order reversal is necessary i.e., if the last line could be

$$(\forall d \in \omega)(Ae \; x_0 \in X_0) \cdots (Ae \; x_{d-1} \in X_{d-1})(\exists \vec{u} \in U_d)[\vec{x} \hat{\;} \vec{u} \in P_j].$$

PROOF. Fix a partition $\mathbf{Q}^\omega = \bigcup_{j<p} P_j$. Trees T_i, $i \in \omega$, will be constructed by induction. Assume that levels $T_i(m)$ have been constructed for $i < n$, $m < n$. Also assume that a set $U_x \subset \mathbf{Q}$ of type η has been associated to each $x \in T_i(m)$. The following facts will be preserved in the induction.

(a) If $m < i$, $T_i(m)$ consists of a single nonrational point. If $i \leqslant m$, $T_i(m)$ is a set of rationals.

(b) If x is rational in $T_i(m)$ then $x \in U_x$.

(c) If $x_1 \leqslant x_2$ in T_i then $U_{x_2} \subset U_{x_1}$.

(d) If $x_1 \in T_{i_1}$ and $x_2 \in T_{i_2}$ for $i_1 \neq i_2$ then $U_{x_1} \cap U_{x_2} = \varnothing$.

(e) If $x \in T_i(m)$, $m < n - 1$, is rational, x has exactly 2 immediate successors, x^- and x^+ in $T_i(m + 1)$.

(f) If $y_1 \in U_{x^-}$ and $y_2 \in U_{x^+}$ then $y_1 <_\mathbf{Q} x <_\mathbf{Q} y_2$. It follows that if $x_1 <_\mathbf{Q} x_2$ and x_1 is T_i incomparable with x_2 then $y_1 <_\mathbf{Q} y_2$ for all $y_1 \in U_{x_1}, y_2 \in U_{x_2}$.

(g) If $x_i \in T_i(m)$, $i \leqslant m$ and $\vec{0}$ is the 0 sequence then for some $j < p$

$$(\text{Ae } y_m \in U_{x_m}) \cdots (\text{Ae } y_0 \in U_{x_0})[\vec{y}\vec{0} \in P_j].$$

(h) Let $\vec{x}$ and j be as in (g). Let $q \leqslant m$, $v \geqslant x_q$ in $T_q(m_q)$ and $y_i \geqslant x_i$ in $T_i(m_i)$ for $q \leqslant i \leqslant m$. Also let the m_i be weakly decreasing with $m_m = m$. Then

$$(\forall w \in U_v)(\text{Ae } z_{q-1} \in Ux_{q-1}) \cdots (\text{Ae } z_0 \in U_{x_0})[\vec{z}w\vec{y}\vec{0} \in P_j].$$

The construction of $T_i(n)$, $i \leqslant n$, proceeds as follows. For $x \in T_i(n - 1)$, $i < n$, let $Ux^+ = \{y \in Ux: x < y\}$ and $Ux^- = \{y \in Ux: y < x\}$. Similarly let $U_n = \mathbf{Q}$. Finitely many successive applications of Laver's theorem with $d = n + 1$ produces η type sets $V_{x^+} \subset U_{x^+}$, $V_{x^-} \subset U_{x^-}$ and $V_n \subset U_n$ with the following property. For every $n + 1$ vector whose ith term has the form $V_{x^\pm}$ or V_n, where $x \in T_i(n - 1)$ when $i < n$, there is a $j < p$ such that

$$(\text{Ae } y_n \in U_n)(\text{Ae } y_{n-1} \in U_{n-1}) \cdots (\text{Ae } y_0 \in U_0)[\vec{y}\vec{0} \in P_j].$$

It can be assumed, by cutting down V_n and the $V_{x^\pm}$'s, that V_n is disjoint from each $V_{x^\pm}$. Properties (a), $\ldots$, (g) will automatically hold if the $x^\pm$, x_n, $U_{x^\pm}$ and U_{x_n} are chosen in such a way that $x^\pm \in U_{x^\pm} \subset V_{x^\pm}$, $x_n \in U_{x_n} \subset V_n$, the $U_{x^\pm}$ and U_{x_n} have type η, and U_{x_n} is assigned to the nonrational $z \in T_n(m)$, $m < n$.

Assume that $x^\pm$ (or x_n) and $U_{x^\pm}$ (or U_{x_n}) have been defined for $x \in T_i(n - 1)$ (or T_n) where $i > q$. Assume further that (hq') holds for all $q' > q$ where:

(hq') Let $m \leqslant n$ and $\vec{U}$ be an m vector in which U_i is U_x, some $x \in T_i(m)$ (or $V_{x^\pm}$ some $x \in T_i(n - 1)$, if $m = n$). Let m_i, $q' \leqslant i \leqslant m$, weakly decrease to m. Let $v \in U_{q'} \cap T_{q'}(mq')$ and $y_i \in U_i \cap T_i(m_i)$, $q' < i \leqslant m$. If $j < p$ is as in (g) for $\vec{U}$ (or as in the definition of the $V_{x^\pm}$ if $m = n$) then

$$(\forall w \in Uv)(\text{Ae } z_{q-1} \in U_{q-1}) \cdots (\text{Ae } z_0 \in U_0)[\vec{z}w\vec{y}\vec{0} \in P_j].$$

$x^\pm$ (or x_n) and $U_{x^\pm}$ (or U_{x_n}) will be defined so as to satisfy (hq). For definiteness assume that $i < n$ and x^+ must be defined. The point of the argument is that each particular $\vec{U}, \vec{y}$ pair necessitates throwing away finitely many elements of V_{x^+}. The resulting U_{x^+} differs finitely from V_{x^+} so it has type η. x^+ is an arbitrary member of the U_{x^+} so constructed.

Let m_i, $q \leqslant i \leqslant m$, $\vec{U}$, and $\vec{y}$ be given as in (hq'). If $m_{q+1} \leqslant n - 1$ then no cutting down of V_{x^+} is necessary because by the inductive assumption of (h) for $n - 1$

$$(\forall w \in Ux)(\text{Ae } z_{q-1} \in U_{q-1}) \cdots (\text{Ae } z_0 \in U_0)\big[\vec{z}\,\hat{}\,w\vec{y}\,\hat{}\,\vec{0} \in P_j\big].$$

If $m_{q+1} = n$ then taking $v = y_{q+1}$ one has by (h$q + 1$)

$$(\text{Ae } z_q \in U_q)(\text{Ae } z_{q-1} \in U_{q-1}) \cdots (\text{Ae } z_0 \in U_0)\big[\vec{z}\vec{y}\,\hat{}\,\vec{0} \in P_j\big].$$

Thus if U_{x^+} avoids the finitely many bad members of U_q one can replace (Ae $z_q \in U_q$) with $(\forall w \in U_{x^+})$. Therefore x^+ and U_{x^+} can be defined for $x \in T_q(n - 1)$ so as to satisfy (hq). Hence when $q = 0$, (h) is satisfied.

Carry out the above definition of the T_i satisfying (a), $\dots$, (h) for each n. It is easy to see that (a'), $\dots$, (f') can be stated for the whole sequence where the primed statement can be obtained from the unprimed one by replacing U_x for $x \in T_i$ with $\text{Succ}(x) = \{y \in T_i: x \leqslant y\}$. The statements (g) and (h) can be replaced by (g').

Let $\vec{x} \in \Pi_{i \in \omega} T_i(n)$. There is a $j < p$ such that

$$(\forall m_n \geqslant n)(\forall y_n \in T_n(m_n) \cap \text{Succ}(x_n))$$

$$(\forall m_{n-1} \geqslant m_n)(\forall y_{n-1} \in T_{n-1}(m_{n-1}) \cap \text{Succ}(x_{n-1})) \cdots$$

$$(\forall m_0 \geqslant m_1)(\forall y_0 \in T_0(m)) \cap \text{Succ}(x_0))\big[\vec{y}\,\hat{}\,\vec{0} \in P_j\big].$$

Partition $\bigcup_{n \in \omega} \Pi_{i \in \omega} T_i(n)$ by sending x to the P_j of (g'). By Theorem 2 there are strong $S_i \subset T_i$ (with level function f) and a single $j < p$ such that

$$(\forall d \in \omega)(\text{Ae } n \in \omega)\bigg(\forall x \in \prod_{i<d} S_i(n)\bigg)\bigg(\exists y \in \prod_{i>d} T_i(f(n))\bigg)\big[\vec{x}\vec{y} \in P_j\big].$$

Let $X_i \subset \mathbf{Q}$ be the set of rationals in S_i. X_i has type η. To show, for example, that X_i is densely ordered consider $x_1 <_{\mathbf{Q}} x_2$ in X_i. Considering the various cases of T_i comparability of x_1 and x_2 one concludes that the S_i successor of either x_1^+ or x_2^- is between x_1 and x_2 (see (f')).

If $d \in \omega$, $\Pi_{i>d} T_i(f(d))$ is finite because all but finitely many $T_i(f(d))$ are singletons. For $\vec{y} \in \Pi_{i>d} T_i(f(d))$ let $\vec{u}(\vec{y}) \in \mathbf{Q}^\omega$ be the sequence defined by $u_j(\vec{y}) = y_{j+d}$ for $j < f(d) - d$ and $u_j(\vec{y}) = 0$ otherwise. Set $U_d = \{\vec{u}(\vec{y}): \vec{y} \in \Pi_{i>d} T_i(f(d))\}$.

It is claimed that

$$(\forall z_{d-1} \in X_{d-1}: \text{level } z_{d-1} \geqslant d)(\forall z_{d-2} \in X_{d-2}: \text{level } z_{d-2} \geqslant \text{level } z_{d-1}) \cdots$$

$$(\forall z_0 \in X_0: \text{level } z_0 \geqslant \text{level } z_1)(\exists \vec{u} \in U_d)\big[\vec{z}\,\hat{}\,\vec{u} \in P_j\big].$$

This implies the desired

$$(\text{Ae } z_{d-1} \in X_{d-1}) \cdots (\text{Ae } z_0 \in X_0)(\exists \vec{u} \in U_d)\big[\vec{z}\,\hat{}\,\vec{u} \in P_j\big].$$

To prove the claim let $z_0, \dots, z_{d-1}$ have decreasing levels above d in the S_i (i.e. above $f(d)$ in the T_i). Let x_i be the predecessor of Z_i in $S_i(d)$. Let $\vec{y} \in \Pi_{i>d} T_i(f(d))$ be such that $\vec{x}\vec{y} \in P_j$. It follows from (g') that $\vec{z}\,\hat{}\,\vec{u}(y) \in P_j$. (Set $m_i = f(d)$, $d \leqslant i \leqslant f(d)$, and use the given $Y_i \in T_i(m_i)$.)

6. Selective ultrafilters and product sacks forcing. We will be concerned with nonprincipal ultrafilters on the power set of ω. Such an ultrafilter can be thought of as a finitely additive 2 valued measure, μ, on subsets of ω such that $\mu(\{n\}) = 0$ when $n \in \omega$ and $\mu(\omega) = 1$. We will often use the terms "small" and "large" to denote sets of μ-measure 0 and 1 respectively. The following is one of several equivalent definitions of selective ultrafilter.

DEFINITION ([1]). μ is *selective* if for every decreasing sequence $X_0 \supset X_1 \supset X_2 \supset \ldots$ of large sets there is a strictly increasing function f from ω to ω with large range such that $(\forall n)[f(n + 1) \in X_{f(n)}]$.

We refer the reader to [6] for set theory in general and for the notions of forcing, Sacks forcing, and product forcing in particular. We assume that Gödel's constructible universe, L, is the ground model but everything done here would work equally well over any universe satisfying the axioms of set theory with dependent choice. As is customary we identify real numbers with subsets of ω through the binary decimal correspondence. The small ambiguity of this correspondence will not concern us here.

Let $\mu \in L$ be an ultrafilter relative to L. If $r \notin L$ is a real number, the family of μ-large sets retains the property of not containing 0 and being closed under finite intersection relative to $L(r)$. It follows that μ generates a 2 valued measure (a filter) on a Boolean subalgebra of $P(\omega)^{L(r)}$. There is no reason in general to expect that μ_r will be an ultrafilter on $P(\omega)^{L(r)}$, and indeed it usually is not. For example if r is a random or a Cohen real, μ_r is undefined on r. A Sacks real is, of course, a minimal constructible degree but even such degrees do not necessarily have any μ_r's which are ultrafilters. If r is a Silver or Gregorieff real, [1], μ_r is defined on r but undefined on some other reals in $L(r)$. (Consider the real t such that $n \in t \leftrightarrow |r \cap n|$ is even.)

R. Solovay, and possibly others, have proven that μ_s is an ultrafilter when s is a Sacks real and μ is selective. In fact μ_s is a selective ultrafilter relative to $L(s)$. Our purpose here is to generalize Solovay's result to d independent (product generic) Sacks reals. Those unfamiliar with the theory of product forcing might wonder why this is a problem. Consider the case of 2 independent Sacks reals, s_0 and s_1. The theory of product forcing says that s_0 is P generic over L and s_1 is P generic over $L(s_0)$ where P is the set of perfect closed subsets of $P(\omega)$ (Sacks conditions) in L. The problem is that P is not the set of Sacks conditions relative to $L(s_0)$ so that s_1, while being Sacks over L, may not be Sacks over $L(s_0)$.

In the following theorem $\mu_{s_0, \ldots, s_{d-1}}$ is the filter generated by μ in $L(s_0, \ldots, s_{d-1})$.

THEOREM 6. *Let S_i, $i < d$, be independent Sacks reals. Let $\mu \in L$ be a selective ultrafilter relative to L. Then $\mu_{s_0, \ldots, s_{d-1}}$ is a selective ultrafilter relative to $L(s_0, \ldots, s_{d-1})$.*

Theorem 6 depends on the following selective ultrafilter version of Theorem 1.

THEOREM 6.1. *Let μ be a selective ultrafilter and let T_i, $i < d$, be trees. Partition $\bigcup_{n \in \omega} \prod_{i < d} T_i(n) = \bigcup_{j < p} P_j$. There is a level family $S_i \subset T_i$, $i < d$, with common level function f and $j < p$ such that $\bigcup_{n \in \omega} \prod_{i < d} S_i(n) \subset P_j$ and Range f is μ-large.*

We will be interested only in the $p = 2$ case of Theorem 6.1. The proof of the general case from the $p = 2$ case is not difficult even if it is also not absolutely immediate.

PROOF OF THEOREM 6 FROM THEOREM 6.1. It will be shown that every real in $L(s_1, \ldots, s_{d-1})$ either includes or is disjoint from a μ-large real in L. Our argument for this from Theorem 6.1 is essentially due to Baumgartner who obtained that every real of $L(s_0, \ldots, s_{d-1})$ either includes or is disjoint from an infinite real in L. Baumgartner used Theorem 1.

The given real is $t(s_0, \ldots, s_{d-1})$ where t is a term in the appropriate forcing language. It suffices to show that if the product condition $(U_0, \ldots, U_{d-1}) \Vdash$ "$t(s'_0, \ldots, s'_{d-1})$ is a real," then some larger (in the componentwise reverse inclusion order) condition $(W_0, \ldots, W_{d-1}) \Vdash$ "$t(s'_0, \ldots, s'_{d-1})$ includes or is disjoint from a μ-large $a \in L$."

Trees T_i, $i < d$, whose nodes are perfect subsets of $P(\omega)$ (Sacks conditions) will now be defined in L. $T_i(0) = \{R_i^*\}$ where $(U_0, \ldots, U_{d-1}) < (U_0^*, \ldots, U_{d-1}^*)$ where $(U_0^*, \ldots, U_{d-1}^*)$ decides "$0 \in t(s_0, \ldots, s_{d-1})$." If the $T_i(n)$ have been defined the $T_i(n + 1)$ will satisfy:

(a) Every node of $T_i(n)$ will have 2 incomparable (i.e. pairwise disjoint) successors in $T_i(n + 1)$.

(b) Every $(V_0, \ldots, V_{d-1}) \in \prod_{i<d} T_i(n + 1)$ decides "$n + 1 \in t(s_0, \ldots, s_{d-1})$."

(c) Each $V \in T_i(n + 1)$ has radius less than $1/2^{n+1}$.

There is no problem producing nodes satisfying (a) and (c). One then steps through them in $(2^{n+1})^d$ steps and makes them satisfy (b).

Partition $\bigcup_{n \in \omega} \prod_{i<d} T_i(n) = P \cup Q$ by saying

$$(V_0, \ldots, V_{d-1}) \in P \cap \prod_{i<d} T_i(n) \leftrightarrow (V_0, \ldots, V_{d-1}) \Vdash \text{"}n \in t(s_0, \ldots, s_{d-1}).\text{"}$$

$$(V_0, \ldots, V_{d-1}) \in Q \cap \prod_{i<d} T_i(n) \leftrightarrow (V_0, \ldots, V_{d-1}) \Vdash \text{"}n \notin t(s'_0, \ldots, s'_{d-1}).\text{"}$$

Let $S_i \subset T_i$, $i < d$, and f be as in Theorem 6.1. Assume, for definiteness, that $\bigcup_{n \in \omega} \prod_{i<d} S_i(n) \subset P$. Let $W_0, \ldots, W_{d-1}$ be the conditions obtained by fusion on the $S_0, \ldots, S_{d-1}$ respectively, i.e.

$$W_i = \bigcap_{n \in \omega} \bigcup S_i(n).$$

Since $S_i(n) \subset T_i(f(n))$ it follows from $\prod_{i<d} S_i(n) \subset P$ that

$$\left(\bigcup S_0(n), \bigcup S_1(n), \ldots, \bigcup S_{d-1}(n) \right) \Vdash \text{"}f(n) \in t(\dot{s}_0, \ldots, \dot{s}_{d-1}).\text{"}$$

Therefore $(W_0, \ldots, W_{d-1}) \Vdash$ "Range $f \subset t(\dot{s}_0, \ldots, \dot{s}_{d-1})$." Since $f \in L$ and Range f is μ large this is what is needed. If one had assumed $\bigcup_{n \in \omega} \prod_{i<d} S_i(n) \subset Q$ one would similarly have obtained $(W_0, \ldots, W_{d-1}) \Vdash$ "Range $f \cap t(\dot{s}_0, \ldots, \dot{s}_{d-1})$ $= \varnothing$.

From the fact that every real of $L(s_0, \ldots, s_{d-1})$ includes or is disjoint from a large real of L it easily follows that $\mu_{s_0, \ldots, s_{d-1}}$ is an ultrafilter relative to $L(s_0, \ldots, s_{d-1})$. To see that it is selective relative to $L(s_0, \ldots, s_{d-1})$ it helps to remark that if $h: \omega \to L$ is in $L(s_0, \ldots, s_{d-1})$ there is a $g: \omega \to L^{<\omega}$ in L such that

for all n, $h(n) \in g(n)$. This is a standard argument due to Sacks [12] for a single Sacks real and presents no problems in the product case. One builds in L the trees T_i, $i < d$, as in the above argument demanding that each $(V_0, \ldots, V_{d-1}) \in \prod_{i<d} T_i(n)$ decides the value of $t(\dot{s}_0, \ldots, \dot{s}_{d-1})$ (now a term for the function h). Fusion over the T_i produces $(W_0, \ldots, W_{d-1})$ which forces $h(n)$ to have one of at most $(2^n)^d$ values. $g(n)$ is the set of these values.

In $L(s_0, \ldots, s_{d-1})$ let $X_0 \supset X_1 \supset X_2 \supset \ldots$ be a decreasing sequence of large sets as in the definition of selective. Let the function $h \in L(s_0, \ldots, s_{d-1})$ be given by $h(n) = Y$ where $Y \in L$, Y is μ large, $Y \subset X_n$, and Y is least in the standard well ordering of L satisfying the foregoing properties. Let g be the function in L where $g: \omega \to L^{<\omega}$ and $h(n) \in g(n)$ for all n. Let $g'(n) = \bigcap g(n)$. $g' \in L$, $g'(n)$ is μ large, and $g'(n) \subset X_n$. Let $Y_n = \bigcap_{i \leqslant n} g'(i)$. The sequence $Y_0 \supset Y_1 \supset Y_2 \supset \ldots$ is in L with each Y_i μ large and $Y_i \subset X_i$. Since μ is selective in L there is an increasing $f \in L$ such that for all n, $f(n+1) \subset Y_{f(n)}$. Since $Y_{f(n)} \subset X_{f(n)}$, f works for the given sequence $X_0 \supset X_1 \supset X_2 \supset \ldots$.

PROOF OF THEOREM 6.1. The hard work for Theorem 6.1 is contained in the following lemma.

LEMMA 1. *Let μ be a nonprincipal ultrafilter and let T_i, $i < d$, be trees. Partition $\bigcup_{n \in \omega} \prod_{i<d} T_i(n) = P \cup Q$. One of the following holds.*
(a)

$$(\exists h)(\forall k)(\exists X \subset \omega \text{ large})(\forall m \in X)\left(\exists M \subset \prod_{i<d} T_i(m)\right)$$

$$[M \subset Q \wedge M \text{ is an } h, k \text{ matrix}].$$

(b)

$$(\forall n)(\exists X \subset \omega \text{ large})(\forall m \in X)\left(\exists M \subset \prod_{i<d} T_i(m)\right)$$

$$[M \subset P \wedge M \text{ is an } n \text{ matrix}].$$

PROOF OF THEOREM 6.1 FROM LEMMA 1. Without loss of generality it may be assumed that alternative (a) of Lemma 1 holds. The argument parallels the proof of Theorem 1 from Theorem 2.1 given in §2.

For each k let X_k be a large set such that for each $m \in X_k$, $Q \cap \prod_{i<d} T_i(m)$ includes an h, $k + 1 - h$ matrix. Without loss of generality it may be assumed that $X_0 \supset X_1 \supset X_2 \supset \ldots$. By the selectivity of μ there is a function $f: \omega \to \omega$ increasing with large range such that $f(k+1) \in X_{f(k)}$. Let X denote the range of f. It follows that if $m \in X$ and $S(X, m)$ is the X-successor of m then $S(X, m) \in X_m$ i.e. $Q \cap \prod_{i<d} T_i(s(X, m))$ includes an h, $m + 1 - h$ matrix. X is, in the notation of §2, a large set of tree levels for the partition.

Following the proof of Theorem 1 from Theorem 2.1 one next obtains a large set of tree levels for the partition all of whose matrices are centered on a single $\vec{a} \in \prod_{i<d} T_i(h)$. Finally, by passing to the appropriate large Z one can be assured that $\vec{a} \in Q$. One then constructs the trees with levels from the large set Z as required for Theorem 6.1.

It should be remarked that the selectivity of μ enters only in the construction of X. If it can be eliminated there one can weaken the hypotheses of Theorem 6.1.

PROOF OF LEMMA 1. This argument parallels that of Halpern and Lauchli [4] except that one also keeps track of the nonprincipal ultrafilter μ. Recall from [4] that a d *word* is a sequence of symbols with length $2d$. The possible symbols in a d word are $\forall a_i$, $\exists A_i$, $\forall x_i$, $\exists x_i$ where $i < d$. The following rules are obeyed in a d word.

(a) $\forall a_i$ and $\exists x_i$ appear in a d word W exactly when $\exists A_i$ and $\forall x_i$ do not appear.

(b) The occurrence of $\exists A_i$ or $\forall a_i$ always precedes the occurrence of $\forall x_i$ or $\exists x_i$.

One writes $W_1 \vdash W_2$, for d words W_1 and W_2, when W_2 can be obtained from W_1 by a finite sequence of the elementary instances of $W_1 \vdash W_2$ listed below. Notation is as follows. α and β stand for a_i, A_i, or x_i. U and V stand for sequences of symbols. $\forall_{i<s} \vec{\alpha}$ or $\exists_{i<s} \vec{\alpha}$ stand for $\forall \alpha_0 \ldots \forall \alpha_{s-1}$ or $\exists \alpha_0 \ldots \exists \alpha_{s-1}$ when $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{s-1})$.

(1) $U \; \forall \alpha \; \forall \beta \; V \vdash U \; \forall \beta \; \forall \alpha \; V$.
(2) $U \; \exists \alpha \; \exists \beta \; V \vdash U \; \exists \beta \; \exists \alpha \; V$.
(3) $U \; \forall a_i \; \exists x_i \; V \vdash U \; \exists A_i \; \forall x_i \; V$.
(4) $U \; \exists A_i \; \forall x_i \; V \vdash U \; \forall a_i \; \exists x_i \; V$.
(5) $\forall_{n\leqslant i<d} \; a_{\sigma(i)} \; \exists_{i<r} \; A_{\sigma(i)} \; U \vdash \exists_{i<r} \; A_{\sigma(i)} \; \forall_{r\leqslant i<d} \; a_{\sigma(i)} \; U$.

The fundamental fact about $\vdash$ is

LEMMA 2 (HALPERN AND LAUCHLI [4]).

$$\forall_{i<d} \vec{a} \; \exists_{i<d} \vec{x} \vdash \exists_{i<d} \vec{A} \; \forall_{i<d} \vec{X}.$$

First assume that the following statement (∗) holds.

$$(\exists n)(\exists_{i<d} \vec{n}: n_i \leqslant n)(\forall p > n)(\forall R \subset \omega \text{ large})(\exists S \subset R \text{ large})$$

$$(\forall m \in S: m \geqslant p)(\exists_{i<d} \vec{B}: B_i \text{ is } p \text{ dense} \wedge B_i \subset T_i(m))$$

$$(\exists_{i<d} \vec{a}: a_i \in T_i(n_i))(\forall_{i<d} \vec{x}: x_i \in B_i \wedge x_i \geqslant a_i)[\vec{x} \in Q].$$

We will argue that alternative (a) of Lemma 1 holds in consequence.

The h of alternative (a) will be the n of (∗). Let k be given. Set $p = h + k$ and $R = \omega$. Let X be the resulting S intersected with $\omega - p$. We must show that $Q \cap \Pi_{i<d} T_i(m)$ includes an (h, k) matrix for all $m \in X$.

Let $\vec{B}$ and $\vec{a}$ be as given in (∗) for $m \in X$. For each $i < d$ let $b_i \in T_i(n)$ be such that $b_i \geqslant a_i$. This is possible since $n_i \leqslant n$. Set $A_i = \{X_i \in B_i: X_i \geqslant b_i\}$. Since $b_i \in T_i(n) = T_i(h)$ and B_i is $p = h + k$ dense it follows that A_i is (h, k) dense. $M = \Pi_{i<d} A_i \subset \Pi_{i<d} T_i(m)$ is an (h, k) matrix. $M \subset Q$ because each $x_i \in A_i$ satisfies $x_i \in B_i$ and $x_i \geqslant b_i \geqslant a_i$.

Now assume that the statement (∗) is false. It remains only to show that alternative (b) of Lemma 1 holds in consequence. The negation of (∗) is the following statement (−∗).

$$(\forall n)(\forall_{i<d} \vec{n}: n_i \leqslant n)(\exists p > n)(\exists R \subset \omega \text{ large})(\forall S \subset R \text{ large})$$

$$(\exists m \in S: m \geqslant p)(\forall_{i<d} \vec{B}: B_i \text{ is } p \text{ dense} \wedge B_i \subset T_i(m))$$

$$(\forall_{i<d} \vec{a}: a_i \in T_i(n_i))(\exists_{i<d} \vec{x}: x_i \in B_i \wedge x_i \geqslant a_i)[\vec{x} \in P].$$

(−∗) can be simplified to the following statement (∗∗).

$$(\forall n)(\forall_{i<d}\ \vec{n}\colon n_i \leqslant n)(\exists p > n)(\exists R \subset \omega - p \text{ large})(\forall m \in R)$$

$$(\forall_{i<d}\ \vec{B}\colon B_i \text{ is } p \text{ dense} \wedge B_i \subset T_i(m))(\forall_{i<d}\ \vec{a}\colon a_i \in T_i(n_i))$$

$$(\exists_{i<d}\ \vec{x}\colon x_i \in B_i \wedge x_i \geqslant a_i)[\vec{x} \in P].$$

For the R in (∗∗) one should use

$$\left\{ m \in R^* - p\colon \left(\bigvee_{i<d}\ \vec{B}\colon B_i \text{ is } p \text{ dense} \wedge B_i \subset T_i(m) \right) \right.$$

$$\left. \left(\bigvee_{i<d}\ \vec{a}\colon a_i \in T_i(n_i) \right)\left(\underset{i<d}{\exists}\ \vec{x}\colon x_i \in B_i \wedge x_i \geqslant a_i \right)[\vec{x} \in P] \right\}$$

where R^* is the R of (−∗). This set is large because, according to (−∗), it has nonzero intersection with every large set.

In order to prove alternative (b) of Lemma 1 one needs not (∗∗) but the following statement (∗∗∗)

$$(\forall n)(\forall_{i<d}\ \vec{n}\colon n_i \leqslant n)(\exists p > n)(\exists R \subset \omega - p \text{ large})(\forall m \in R)$$

$$(\forall_{i<d}\ \vec{B}\colon B_i \text{ is } p \text{ dense} \wedge B_i \subset T_i(m))(\exists_{i<d}\ \vec{A}\colon A_i \subset B_i, n_i \text{ dense})$$

$$(\forall_{i<d}\ \vec{x}\colon x_i \in A_i)[\vec{x} \in P].$$

To derive alternative (b) from (∗∗∗) let n be as given in alternative (b), let $\vec{n} = (n, n, \dots, n)$, and let X be the R from (∗∗∗). For each $m \in X$ let $B_i = T_i(m)$, all $i < d$ and let A_i be as obtained in (∗∗∗). $\Pi_{i<d} A_i$ is an n matrix and, according to (∗∗∗), $\Pi_{i<d} A_i \subset P$.

Thus Lemma 1 has been reduced to proving (∗∗∗) from (∗∗). This is where the d words are used. For each d word W let $\Phi(W)$ be the statement

$$(\forall n)(\forall_{i<d}\ \vec{n}\colon n_i \leqslant n)(\exists p > n)(\exists R \subset \omega - p \text{ large})(\forall m \in R)$$

$$(\forall_{i<d}\ \vec{B}\colon B_i \text{ is } p \text{ dense} \wedge B_i \subset T_i(m))W[\vec{x} \in P]$$

where the quantifiers in W are interpreted as follows.

$\forall a_i$ is interpreted as $(\forall a_i \in T_i(n_i))$,

$\exists A_i$ is interpreted as $(\exists A_i\colon A_i \subset B_i, n_i \text{ dense})$,

$\exists x_i$ is interpreted as $(\exists x_i\colon x_i \in B_i \wedge x_i \geqslant a_i)$,

$\forall x_i$ is interpreted as $(\forall x_i\colon x_i \in A_i)$.

These interpretations make sense because of the rules about occurrences of the quantifiers in a d word.

LEMMA 3. *If $W_1 \vdash W_2$ then $\Phi(W_1) \Rightarrow \Phi(W_2)$.*

Since (∗∗) is $\Phi(\forall_{i<d}\ \vec{a}\ \exists_{i<d}\ \vec{x})$ and (∗∗∗) is $\Phi(\exists_{i<d}\ \vec{A}\ \forall_{i<d}\ \vec{x})$ Lemma 2 and Lemma 3 complete the proof of (∗∗∗) from (∗∗) and therefore the proof of Lemma 1.

PROOF OF LEMMA 3. It suffices to show the 5 special cases of the lemma corresponding to the 5 defining instances of $\vdash$. Instances 1 and 2 are immediate while instances 3 and 4 amount to the observation that there is an n_i dense $A_i \subset B_i$ such that $(\forall x_i \in A_i)[\ \dots\ ]$ exactly when for each $a_i \in T_i(n_i)$ there is an $x_i \in B_i$, such that $x_i \geqslant a_i \wedge [\ \dots\ ]$. Instance 5 of the definition of $\vdash$ must be treated with some care.

By renumbering the trees it suffices to assume that $W_1 = \forall_{r \leqslant i < d} \, \vec{a} \, \exists_{i < r} \, \vec{A} U$ and $W_2 = \exists_{i < r} \, \vec{A} \, \forall_{r \leqslant i < d} \, \vec{a} U$. Let n and $\vec{n}$ be given as in $\Phi(W_2)$. Let N be the cardinality of the product $\Pi_{r \leqslant i < d} \, T_i(n_i)$. For $j < N$ define p_j and R_j as follows. p_0 and R_0 are those produced by $\Phi(W_1)$ for the given n and $\vec{n}$. p_{j+1} and R_{j+1} are those produced by $\Phi(W_1)$ for $n' = p_j$ and $\vec{n}$ in which $n_i' = p_j$ when $i < r$ and $n_i' = n_i$ when $r \leqslant i < d$. p is taken to be p_{N-1} while R is taken to be $\bigcap_{j < N} R_j$. It must now be verified that

$$(\forall m \in R)\Big(\underset{i < d}{\forall} \, \vec{B}: B_i \text{ is } p \text{ dense} \wedge B_i \subset T_i(m)\Big)\Big(\underset{i < r}{\exists} \, \vec{A}\Big)\Big(\underset{r \leqslant i < d}{\forall} \, \vec{a}\Big) U[\vec{x} \in P]$$

holds under the designated interpretations of the quantifiers in W_2.

Accordingly, let $m \in R$ and $B_i \subset T_i(m)$ p dense be given for all $i < d$. $\vec{A}_j, j < N$ is an r vector defined by downward induction on j. Let $a_j: j < N$ be an enumeration of $\Pi_{r \leqslant i < d} \, T_i(n_i)$. $\vec{A}_{N-1}$ is such that $A_{N-1,i} \subset B_i$ is p_{N-2} dense, $i < r$, and $U(\vec{B}, \vec{A}_{N-1}, \vec{a}_{N-1})[\vec{x} \in P]$ holds. $\vec{A}_{N-1}$ exists because $m \in R \subset R_{N-1}$ and p_{N-1}, R_{N-1} solve $\Phi(W_1)$ for the $\vec{n}$ which is identically p_{N-2} for $i < r$ and agrees with $\vec{n}$ otherwise.

Similarly for $0 \leqslant j < N - 1$, $\vec{A}_j$ is such that $A_{j,i} \subset A_{j+1,i}$ is p_{j-1} dense and $U(\vec{B}_{j+1}, \vec{A}_j, \vec{a}_j)[\vec{x} \in P]$ holds where $B_{j+1} = A_{j+1,i}$ for $i < r$ and $B_{j+1} = B_i$ for $r \leqslant i < d$. As above, A_j exists because $m \in R \subset R_j$ and p_j, R_j solve $\Phi(W_1)$ for the $\vec{n}'$ which is identically p_{j-1} for $i < r$ and agrees with $\vec{n}$ otherwise.

We remark that if $j < k \leqslant N - 1$ then $U(\vec{B}_{j+1}, \vec{A}_j, \vec{a}_k)[\vec{x} \in P]$ also holds. If $i < r$ then x_i occurs in U bound in the quantifier $(\forall x_i: x_i \in A_i)$. When this is applied to $A_{j,i}$ instead of $A_{k,i}$ the effect is only to restrict the domain of universal quantification because $A_{j,i} \subset A_{k,i}$. Thus the statement continues to hold. If $r \leqslant i < d$ then x_i occurs in U bound in the quantifier $(\exists x_i: x_i \in B_{j+1,i} \wedge x_i \geqslant a_i)$. $B_{j+1,i} = B_{k+1,i} = B_i$ for $r \leqslant i < d$ so no change in the domain of quantification occurs.

The construction is varied at the last i.e. $j = 0$ step so that $A_{j,i}$ is n_i dense. It follows that $\vec{A}_0$ satisfies $A_{0,i} \subset B_i$ and $U(\vec{B}_1, \vec{A}_0, \vec{a}_k)[\vec{x} \in P]$ holds for every $0 < k < N$. Since B_i only occurs in U for $r \leqslant i < d$ one has $U(\vec{B}, \vec{A}_0, \vec{a})[\vec{x} \in P]$ holds for the original $\vec{B}$ and every $\vec{a} \in \Pi_{r \leqslant i < d} \, T_i(n_i)$.

Therefore using $\vec{A}_0$ as the required $\vec{A}$ one has

$$\underset{i < r}{\exists} \, \vec{A} \, \underset{r \leqslant i < d}{\forall} \, \vec{a} \, U(\vec{B}, \vec{A}, \vec{a})[\vec{x} \in P]$$

holds. This completes the proof of Lemma 3, hence of Lemma 1, Theorem 6.1, and Theorem 6.

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